Markscheme

May 2015

Calculus

Higher level

Paper 3

12 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking May 2015**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	Further working seen	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**).*
*A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. $f(0) = 0$ ***A1***

$f'(x) = -e^{-x}\cos x - e^{-x}\sin x + 1$ ***M1A1***

$f'(0) = 0$ ***(M1)***

$f''(x) = 2e^{-x}\sin x$ ***A1***

$f''(0) = 0$

$f^{(3)}(x) = -2e^{-x}\sin x + 2e^{-x}\cos x$ ***A1***

$f^{(3)}(0) = 2$

the first non-zero term is $\frac{2x^3}{3!}\left(=\frac{x^3}{3}\right)$ ***A1***

Note: Award no marks for using known series.

[7 marks]

2. (a) **METHOD 1**

$\frac{dy}{dx} = -\frac{1}{x^2}\int f(x)\,dx + \frac{1}{x}f(x)$ ***M1M1A1***

$x\frac{dy}{dx} + y = f(x),\ x > 0$ ***AG***

Note: ***M1*** for use of product rule, ***M1*** for use of the fundamental theorem of calculus, ***A1*** for all correct.

METHOD 2

$x\frac{dy}{dx} + y = f(x)$

$\frac{d(xy)}{dx} = f(x)$ ***(M1)***

$xy = \int f(x)dx$ ***M1A1***

$y = \frac{1}{x}\int f(x)dx$ ***AG***

[3 marks]

continued…

Question 2 continued

(b) $y = \frac{1}{x}\left(2x^{\frac{1}{2}} + c\right)$ ***A1A1***

> **Note:** ***A1*** for correct expression apart from the constant, ***A1*** for including the constant in the correct position.

attempt to use the boundary condition ***M1***

$c = 4$ ***A1***

$y = \frac{1}{x}\left(2x^{\frac{1}{2}} + 4\right)$ ***A1***

[5 marks]

> **Note:** Condone use of integrating factor.

Total [8 marks]

3. (a) **METHOD 1**

$(0<)\frac{1}{n^2 \ln(n)} < \frac{1}{n^2}$, (for $n \ge 3$) ***A1***

$\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges ***A1***

by the comparison test ($\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges implies) $\sum_{n=2}^{\infty} \frac{1}{n^2(\ln n)}$ converges ***R1***

> **Note:** Mention of using the comparison test may have come earlier. Only award ***R1*** if previous 2 ***A1***s have been awarded.

METHOD 2

$\lim_{n\to\infty}\left(\frac{\frac{1}{n^2 \ln n}}{\frac{1}{n^2}}\right) = \lim_{n\to\infty}\frac{1}{\ln n} = 0$ ***A1***

$\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges ***A1***

by the limit comparison test (if the limit is 0 and the series represented by the denominator converges, then so does the series represented by the

continued…

Question 3 continued

numerator, hence) $\sum_{n=2}^{\infty} \frac{1}{n^2(\ln n)}$ converges ***R1***

Note: Mention of using the limit comparison test may come earlier. Do not award the ***R1*** if incorrect justifications are given, for example the series "converge or diverge together". Only award ***R1*** if previous 2 ***A1***s have been awarded.

[3 marks]

(b) (i) **EITHER**

$\ln(n) + \ln\left(1 + \frac{1}{n}\right) = \ln\left(n\left(1 + \frac{1}{n}\right)\right)$ ***A1***

OR

$\ln(n) + \ln\left(1 + \frac{1}{n}\right) = \ln(n) + \ln\left(\frac{n+1}{n}\right)$

$= \ln(n) + \ln(n+1) - \ln(n)$ ***A1***

THEN

$= \ln(n+1)$ ***AG***

(ii) attempt to use the ratio test $\frac{n}{(n+1)} \frac{\ln(n)}{\ln(n+1)}$ ***M1***

$\frac{n}{n+1} \to 1$ as $n \to \infty$ ***(A1)***

$\frac{\ln(n)}{\ln(n+1)} = \frac{\ln(n)}{\ln(n) + \ln\left(1 + \frac{1}{n}\right)}$ ***M1***

$\to 1$ (as $n \to \infty$) ***(A1)***

$\frac{n}{(n+1)} \frac{\ln(n)}{\ln(n+1)} \to 1$ (as $n \to \infty$) hence ratio test is inconclusive ***R1***

Note: A link with the limit equalling 1 and the result being inconclusive needs to be given for ***R1***.

[6 marks]

(c) (i) consider $f(x) = \frac{1}{x \ln x}$ (for $x > 1$) ***A1***

$f(x)$ is continuous and positive ***A1***

and is (monotonically) decreasing ***A1***

Note: If a candidate uses n rather than x, award as follows $\frac{1}{n \ln n}$ is positive and decreasing ***A1A1*** $\frac{1}{n \ln n}$ is continuous for $n \in \mathbb{R}$, $n > 1$ ***A1*** (only award this mark if the domain has been explicitly changed).

continued…

Question 3 continued

(ii) consider $\int_2^R \frac{1}{x \ln x} \mathrm{d}x$ ***M1***

$= \left[\ln(\ln x)\right]_2^R$ ***(M1)A1***

$\to \infty$ as $R \to \infty$ ***R1***

hence series diverges ***A1***

Note: Condone the use of ∞ in place of R.

Note: If the lower limit is not equal to 2, but the expression is integrated correctly award ***M0M1A1R0A0***.

[8 marks]

Total [17 marks]

4. (a) $\lim_{x\to\infty}\frac{x^2}{e^x} = \lim_{x\to\infty}\frac{2x}{e^x}$ ***M1A1***

$\lim_{x\to\infty}\frac{2}{e^x} = 0$ ***M1A1***

Note: Award ***M1*** for an attempt at differentiating for a second time.

[4 marks]

(b) attempt to integrate by parts ***M1***

$\int x^2 e^{-x} \mathrm{d}x = -x^2 e^{-x} + \int 2x e^{-x} \mathrm{d}x$ ***(A1)***

$= -x^2 e^{-x} - 2x e^{-x} + \int 2 e^{-x} \mathrm{d}x$ ***(A1)***

$= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} \ (+c)$ ***A1***

$\int_0^R x^2 e^{-x} \mathrm{d}x = -R^2 e^{-R} - 2R e^{-R} - 2e^{-R} + 2$ ***M1A1***

$\lim_{R\to\infty}\left(\int_0^R x^2 e^{-x} \mathrm{d}x\right) = 2$ ***M1A1***

Note: Award ***M1*** for consideration of the limit and ***A1*** for correct limiting value.

hence the improper integral converges ***AG***

Note: Do not award the final four marks to candidates who do not consider *R*.

[8 marks]

Total [12 marks]

5. (a) (i) $f'(x) = 3x^2 + 6x$ ***A1***

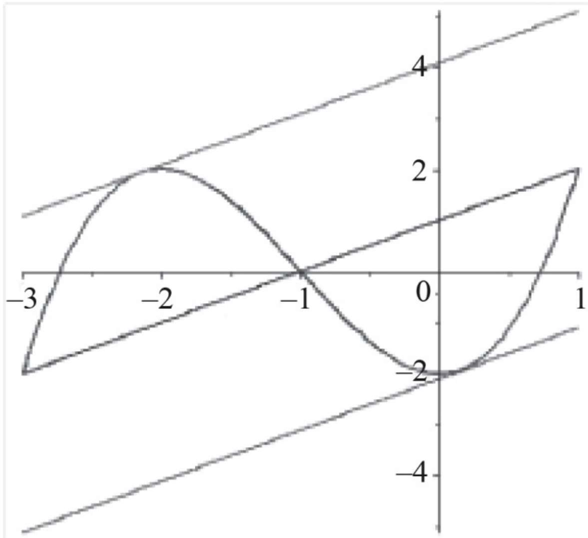
gradient of chord $= 1$ ***A1***

$3c^2 + 6c = 1$

$c = \dfrac{-3 \pm 2\sqrt{3}}{3}$ $(= -2.15,\ 0.155)$ ***A1A1***

Note: Accept any answers that round to the correct 2sf answers $(-2.2,\ 0.15)$.

(ii)



award ***A1*** for correct shape and clear indication of correct domain, ***A1*** for chord (from $x = -3$ to $x = 1$) and ***A1*** for two tangents drawn at their values of c ***A1A1A1***

[7 marks]

(b) (i) **METHOD 1**

(if a theorem is true for the interval $[a, b]$, it is also true for any interval $[x_1, x_2]$ which belongs to $[a, b]$)

suppose $x_1, x_2 \in [a, b]$ ***M1***

by the MVT, there exists c such that $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$ ***M1A1***

hence $f(x_1) = f(x_2)$ ***R1***

as x_1, x_2 are arbitrarily chosen, $f(x)$ is constant on $[a, b]$

Note: If the above is expressed in terms of a and b award ***M0M1A0R0***.

METHOD 2

(if a theorem is true for the interval $[a, b]$, it is also true for any interval $[x_1, x_2]$ which belongs to $[a, b]$)

suppose $x \in [a, b]$ ***M1***

continued…

Question 5 continued

by the MVT, there exists c such that $f'(c) = \dfrac{f(x) - f(a)}{x - a} = 0$ ***M1A1***

hence $f(x) = f(a) =$ constant ***R1***

(ii) attempt to differentiate $(x) = 2\arccos x + \arccos\left(1 - 2x^2\right)$ ***M1***

$-2\dfrac{1}{\sqrt{1-x^2}} - \dfrac{-4x}{\sqrt{1-\left(1-2x^2\right)^2}}$ ***A1A1***

$= -2\dfrac{1}{\sqrt{1-x^2}} + \dfrac{4x}{\sqrt{4x^2-4x^4}} = 0$ ***A1***

Note: Only award ***A1*** for 0 if a correct attempt to simplify the denominator is also seen.

$f(x) = f(0) = 2 \times \dfrac{\pi}{2} + 0 = \pi$ ***A1AG***

Note: This ***A1*** is not dependent on previous marks.

Note: Allow any value of $x \in [0, 1]$.

[9 marks]

Total [16 marks]